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Ward's Identity in Critical Dynamics

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We consider the relaxation of an order-parameter fluctuation of wave number k in a system undergoing a second-order phase transition. In general, close to the critical point, where $k^{-1} \ll \kappa^{-1}$ (the correlation length) the relaxation rate has a linear dependence on κ/k of the form $\gamma(k, \kappa) = \gamma(k, 0) \times (1 - a\kappa/k)$. In analogy with the use of Ward's identity in elementary particle physics, we show that the numerical coefficient a is readily calculated by means of a "mass insertion." We demonstrate, furthermore, that this initial linear drop is the main feature of the full κ/k dependence of the scaling function $R^{-x}\gamma(k, \kappa)$, where x is the dynamic critical exponent and $R = (k^2 + \kappa^2)^{1/2}$ is the "distance" variable.

KEY WORDS: Ward's identity; critical dynamics; scaling function; phase transition.

1. INTRODUCTION

One of the central tasks in the theory of critical dynamics is the computation of the rate $\gamma(k, \kappa)$ of a relaxing mode of wave number k at a temperature for which the correlation length is κ^{-1}. As κ is a monotonic function of the temperature, it serves to describe the critical temperature dependence. At the critical point, $\kappa = 0$, so that the problem simplifies to the determination of the function $\gamma(k, 0)$ of the single variable k. According to the theory of dynamic scaling[1,2] this function is generally of the form k^x, where x is a critical exponent. The determination of $\gamma(k, \kappa)$ in the full quadrant of the κ–k plane shown in Fig. 1 is a much more complicated problem, requiring in general the solution of coupled integral equations.

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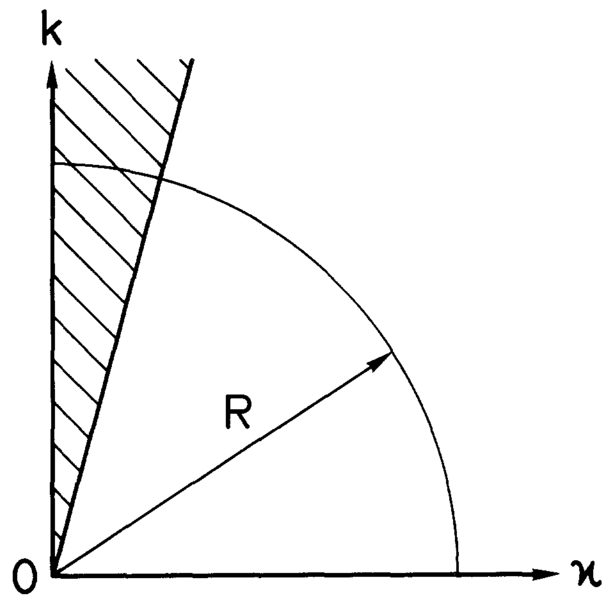

Fig. 1. General features of the dependence of a critical relaxation rate on wave number, k, and correlation length, κ^{-1}. Natural variables are the "radius" $R=(k^2+\kappa^2)^{1/2}$, and the ratio, or "angle variable," κ/k. The linear dependence on κ/k for $\kappa \ll k$ (shaded region) is readily calculated by the Ward's identity method.

The purpose of the present paper is to demonstrate, in spite of the intrinsic mathematical complexity of the problem, the following two points:

1. In many cases the overall behavior of $\gamma(k, \kappa)$ is determined largely by its first departure from $\gamma(k, 0)$, which is linear in κ/k for $\kappa \ll k$. This linear region is shown by the shaded sector in Fig. 1.

2. This first departure can be calculated in a simple way by an application of Ward's identity, in the form in which it is defined below.

For any given problem the approach that we will present here does not take the place of an exact numerical solution of the underlying coupled integral equations. But on the other hand, the possibility of a computer solution does not make the approach advocated here superfluous. The two treatments complement one another. It is often stated that one should not put a problem on the computer unless one already knows beforehand the answer—at least, qualitatively. This is exactly the role of the Ward's identity approach. It can serve as a first approximation for a computer iteration of the coupled integral equations. Furthermore, this first approximation may in many cases represent a solution of adequate accuracy. This is because of the inherent approximations in the integral equations them-

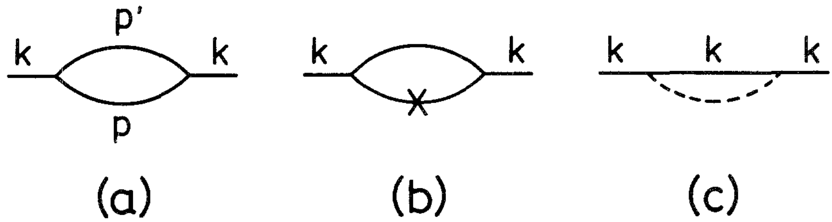

Fig. 2. Ward's identity method. Differentiating the convolution integral represented by (a) results in the "mass insertion" shown by the cross in (b). Because of the predominant contribution of the small momenta the intermediate line carrying the mass insertion can be replaced by the "spectator" line [dashed in (c)] carrying zero momentum.

selves, which can cause the added accuracy of the computer solution to be of questionable significance, or, possibly, even irrelevant.

In Section 2 below we introduce the use of Ward's identity and the so-called "mass insertion" in the simple context of a static correlation function. Section 3 deals with the more complicated situation that obtains in critical dynamics. Section 4 illustrates the general approach by a specific application to the ferromagnetic phase transition.[3] This section concludes with a comparison of the dynamic scaling function obtained by the Ward's identity method with that found by Résibois and Piette.[(4)] Section 5 constitutes a brief summary.

2. MASS INSERTION

To demonstrate the main two points listed above it will suffice to consider a typical single-loop contribution to $\gamma(k, \kappa)$, as shown by Fig. 2. The integration over the internal wave numbers or "momenta," $\mathbf{p}$ and $\mathbf{p}'$, is to be carried out under the conservation of momentum constraint $\mathbf{p}+\mathbf{p}'=\mathbf{k}$. The internal lines in Fig. 2a represent the Fourier transform of the order-parameter correlation function. Because of the small value of the anomalous dimension exponent[(5)] η it is generally sufficiently accurate for critical dynamics to set it equal to zero and to use the order-parameter correlation function in the form of the Ornstein–Zernike approximation

$$G(r, \kappa)=\frac{1}{4\pi r}e^{-\kappa r} \tag{2.1}$$

where r is the spatial separation of the two interaction points shown in Fig. 2a. The first departure from the critical point is given by the derivative

[3] For other examples of this method see Ref. 3.

with respect to κ of the integral represented by Fig. 2a. Carrying out the differentiation inside the integral involves replacing G wherever it occurs by

$$\left.\frac{\partial G}{\partial \kappa}\right|_{\kappa=0} = -\frac{1}{4\pi} \tag{2.2}$$

Equation (2.2) exhibits the central point of this paper, which is that the complete disappearance of r from Eq. (2.2) greatly simplifies the computation of $\partial\gamma(k, \kappa)/\partial\kappa\,|_{\kappa=0}$. The same result can be obtained from the Fourier transform of G,

$$g(p, \kappa) = \frac{1}{p^2+\kappa^2} \tag{2.3}$$

and its derivative

$$\frac{\partial g}{\partial \kappa} = -2\kappa g^2 \tag{2.4}$$

This is equivalent to the famous "Ward's identity" of quantum electrodynamics,[(6)] being an equality between a vertex (right-hand side) and a "mass" derivative (left-hand side). Equation (2.4) is illustrated by the cross in Fig. 2b, indicating a "mass insertion." The order-parameter correlation function is thereby converted into a zero-momentum vertex function, as we now explain. Because of the strong p dependence of Eq. (2.4) the p integration is concentrated in the immediate vicinity of the origin. The remaining factors of the integrand can therefore be evaluated at $p=0$ and taken outside of the integral, leaving

$$\begin{aligned}\frac{\partial}{\partial \kappa}\frac{1}{8\pi^3}\int d^3p\, g(p, \kappa) &= \frac{-\kappa}{\pi^2}\int_0^\infty \frac{p^2\,dp}{(p^2+\kappa^2)^2} \\ &= -\frac{\kappa}{\pi^2}\cdot\frac{\pi}{4\kappa} = -\frac{1}{4\pi}\end{aligned} \tag{2.5}$$

This alternative derivation of Eq. (2.2) can be interpreted as the replacement of $g(p, \kappa)$ by the dashed "spectator" line in Fig. 2c. A spectator line carries no momentum and has the strength $-1/4\pi$. Thus every n-loop integral is reduced by the κ differentation to a sum of $(n-1)$-loop integrals. The fact that the integrands of these $(n-1)$-loop integrals are functions only of k is a further convenience in their evaluation.

For illustrating the method described above it suffices to consider the single-loop polarization integral[4]

$$C(k, \kappa) = \int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}} G^2(r, \kappa) \tag{2.6}$$

[4] This integral plays a central role in the screening theory, or n^{-1} expansion. See, e.g., Ref. 7.

Although this is not a dynamic quantity, it does offer a prototype for the application of Ward's identity. Substituting Eq. (2.1) and integrating over angle reduces Eq. (2.6) to

$$C(k, \kappa) = \frac{1}{4\pi k} \int_0^\infty \frac{dr}{r} \sin kre^{-2\kappa r} \tag{2.7}$$

The appearance of 2κ in the integrand of this integral has led to the statement of the general "rule-of-thumb" concerning the equivalence[8] between k and 2κ. Explicit evaluation of Eq. (2.7) yields

$$C(k, \kappa) = \frac{1}{4\pi k} \tan^{-1} \frac{k}{2\kappa} \tag{2.8}$$

with the limiting values

$$C(k, 0) = \frac{1}{8k} \tag{2.9}$$

and

$$C(0, \kappa) = \frac{1}{8\pi\kappa} \tag{2.10}$$

For exhibiting explicitly the behavior of $C(k, \kappa)$ in the region $\kappa \ll k$ (shaded region of Fig. 1) we rewrite Eq. (2.8) as

$$\begin{aligned} C(k, \kappa) &= \frac{1}{8k} - \frac{1}{4\pi k} \tan^{-1} \frac{2\kappa}{k} \\ &\simeq \frac{1}{8k} - \frac{1}{2\pi} \frac{\kappa}{k^2} \\ &= \frac{1}{8k}\left(1 - a\frac{\kappa}{k}\right) \end{aligned} \tag{2.11}$$

where

$$a = \frac{4}{\pi} \tag{2.12}$$

As a demonstration of Ward's identity method we now derive Eqs. (2.11) and (2.12) without the benefit of the full κ dependence, which is in general not available. In momentum space, Eq. (2.6) becomes the convolution integral

$$C(k, \kappa) = \frac{1}{8\pi^3} \int d^3p g(p, \kappa)\, g(p', \kappa) \tag{2.13}$$

with the conservation of momentum constraint

$$\mathbf{p} + \mathbf{p}' = \mathbf{k} \tag{2.14}$$

as mentioned above. Differentiating with respect to κ, substituting Eq. (2.5), and carrying out the integration in the vivinity of $p = 0$ and $p' = 0$ yields

$$\left.\frac{\partial C(k, \kappa)}{\partial \kappa}\right|_{\kappa=0} = 2 \cdot \frac{-1}{4\pi} g(k, 0) = \frac{-1}{2\pi k^2} \tag{2.15}$$

the factor of 2 coming from the fact that either of the two lines of the "bubble" graph of Fig. 1 can become the spectator line. The desired linear coefficient then follows from the definition

$$a = \frac{-k}{C(k, 0)} \left.\frac{\partial C(k, \kappa)}{\partial \kappa}\right|_{\kappa=0} = \frac{4}{\pi} \tag{2.16}$$

in agreement with Eq. (2.12).

Because $C(k, \kappa)$ is a homogeneous function of its two variables it is convenient to reexpress it in terms of the ratio κ/k and the "distance" in the κ–k plane,

$$R(k, \kappa) \equiv (k^2 + \kappa^2)^{1/2} \tag{2.17}$$

These are effectively the angle and radial variables, respectively, of a circular polar coordinate system. Along the two limiting Cartesian axes we have

$$C(k, 0) = \frac{1}{8R} \tag{2.18}$$

and

$$C(0, \kappa) = \frac{1}{8\pi R} \tag{2.19}$$

A scaling function depending only on the "angle" variable κ/k can therefore be defined as

$$S\left(\frac{\kappa}{k}\right) \equiv 8R(k, \kappa)\, C(k, \kappa) \tag{2.20}$$

This function describes the variation of $C(k, \kappa)$ along the arc of the circle shown in Fig. 1 at $R = \text{const}$. The limiting values at the two ends of the arc, according to Eqs. (2.18) and (2.19), are

$$S(0) = 1 \tag{2.21}$$

and

$$S(\infty)=\frac{1}{\pi} \tag{2.22}$$

Substitution of Eqs. (2.8) and (2.17) into Eq. (2.20) gives

$$S\left(\frac{\kappa}{k}\right)=\frac{2}{\pi}\left(1+\frac{\kappa^2}{k^2}\right)^{1/2}\tan^{-1}\frac{k}{2\kappa} \tag{2.23}$$

which is plotted vs. κ/k as the dot–dash curve in Fig. 3. The dashed slanting straight line shows the approximation

$$S\left(\frac{\kappa}{k}\right)\simeq 1-a\frac{\kappa}{k} \tag{2.24}$$

with $a=4/\pi$ from Eq. (2.12). Although Eq. (2.24) is valid only for $\kappa/k \ll 1$, it is evident in Fig. 3 that this linear approximation accounts qualitatively for a major portion of the drop in the scaling function upon going away from the critical point. In order to demonstrate the full usefulness of the

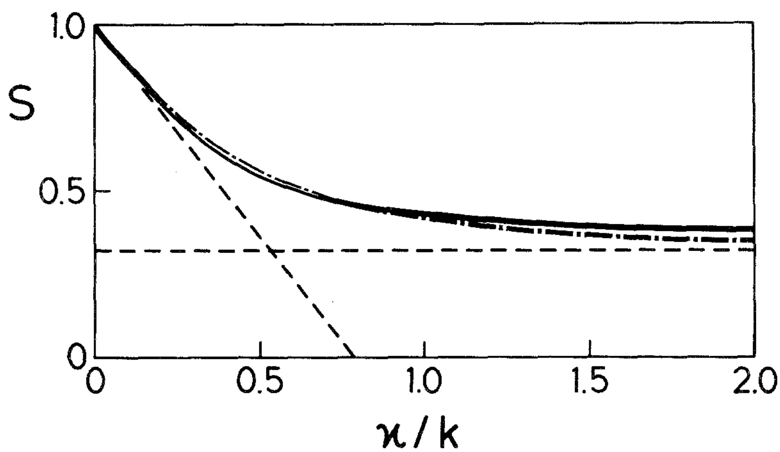

Fig. 3. Scaling function for single-loop polarization graph of Fig. 2a vs. κ/k, the reciprocal of the product of wave number times correlation length. The solid curve shows $\tilde{S}$, the two-parameter approximant, with the parameters fixed by Ward's identity (initial slope shown by slanting dashed line) and by the "rule-of-thumb" value for the $\kappa/k \to \infty$ limit. The exact $\kappa/k \to \infty$ limit is indicated by the horizontal dashed line. The close agreement between the dot–dash curve (S, the exact scaling function) and the solid curve ($\tilde{S}$, the approximant) demonstrates the usefulness of a two-parameter approximant in cases where an exact expression is not available.

Ward's identity method we present in Appendix A a simple method for obtaining an approximate scaling function $\tilde{S}(\kappa/k)$, as illustrated by the solid curve in Fig. 3.

3. CRITICAL DYNAMICS

In this section we discuss, in a somewhat formal and abstract fashion, a complication that enters when we apply Ward's identity approach to a particular problem in critical dynamics. In Section 4 we study the same point in more concrete form in the specific case of the ferromagnet. In general we have to deal with n different relaxation rates $\gamma_i(k, \kappa)$, for $i = 1, 2,..., n$. Our task, in analogy with Eq. (2.16), is to determine the numerical values of the n different linear coefficients

$$a_i = -\frac{k}{\gamma_i(k, 0)} \frac{\partial}{\partial \kappa} \gamma_i(k, \kappa) \mid_{\kappa=0} \tag{3.1}$$

But these coefficients are *not* given directly by the mass insertions discussed in the preceding section. The mass insertions affect only the "static," or equal-time order-parameter correlation functions $g(p, \kappa)$, and thereby produce the coefficients

$$b_i = -\frac{k}{\gamma_i(k, 0)} \frac{\partial}{\partial \kappa} \gamma_i(k, \kappa) \mid_{\kappa=0}^{\text{static}} \tag{3.2}$$

The superscript denotes that, in the integrals specifying $\gamma_i(k, \kappa)$, only the direct changes produced by the κ dependence of the static correlation function factors in the integrands are to be included in b_i. Thus the b_i are the true expression of the mass insertion effect studied in the preceding section. The a_i, on the other hand, require additional contributions because of the coupling of the ith mode with the other relaxing modes of the system, which we denote by $j = 1, 2,..., n$. The rates $\gamma_j(k, \kappa)$ occur in the denominators of the integrals that specify $\gamma_i(k, \kappa)$. Therefore the linear variation of the $\gamma_j(k, \kappa)$, for $\kappa \ll k$, produces an additional linear variation in $\gamma_i(k, \kappa)$, which can be written in the form

$$\Delta a_i = -\sum_{j=1}^{n} K_{ij} a_j \tag{3.3}$$

where the K_{ij} are certain well-defined convolution integrals to be evaluated for $\kappa = 0$. The full linear coefficient is consequently $a_i = b_i - \Delta a_i$, which it is convenient to write in matrix form as

$$a = b - Ka \tag{3.4}$$

whose matrix solution is

$$a = (I + K)^{-1} b \tag{3.5}$$

where I is the identity matrix.

4. FERROMAGNET

As a concrete example of the general method discussed in the previous section we sketch here the determination of the dynamic scaling function for the isotropic Heisenberg ferromagnet in the vicinity of its Curie point. Our interest is to illustrate the procedure, instead of producing exact numerical results. Therefore, in simplifying the exposition, we do not hesitate to make rough approximations, provided that these do not change the picture in any essential way. The precise numerical details can be found in Ref. 3. From the decoupled-mode theory,[8,9] the relaxation rate for a fluctuation of wave number, or "momentum," k, is, measured in a certain convenient unit,

$$\gamma(k, \kappa) = \frac{g^{-1}(k, \kappa)}{8\pi^3} \int \frac{d^3p(p^2 - p'^2)^2\, g(p, \kappa)\, g(p', \kappa)}{\gamma(k, \kappa) + \gamma(p, \kappa) + \gamma(p', \kappa)} \tag{4.1}$$

The convolution integral of Eq. (4.1) illustrates the theme of this paper, which is that the critical temperature dependence for $\kappa \ll k$ has its origin entirely in the divergence of the static, or equal-time correlation function $g(p, \kappa)\,|_{p=0} \propto k^{-2+\eta} \cong k^{-2}$. This $p = 0$ divergence influences a region of p space of radius of $O(k)$, so that its total effect is proportional to $k^3 k^{-2} = k$. The validity of this result depends upon neglecting the anomalous dimension exponent η, which involves an acceptable error of the order[5] of 4%. More serious is the approximation of $g(p, \kappa)$ by the Ornstein–Zernike formula of Eq. (2.3), which neglects the Fisher–Langer tail. The latter changes the κ dependence of $g(p, \kappa)$ for p values which exceed κ by an order of magnitude. [For $p = O(\kappa)$ Eq. (2.3) remains a good approximation.] The sum rule[5] requires that $\partial/\partial\kappa \int d^3p g(p, \kappa)$ have a κ dependence of the form of a sum of terms κ^y, where y equals $1/\nu - 1$ and $(1-\alpha)/\nu - 1$. This rigorous result, standing in contradiction to the constant value given in Eq. (2.5), indicates a definite breakdown of our computational method in the region $0 < \kappa \ll 0.1k$. The method, however, recovers its validity for larger values of κ. This is because the convolution integral of Eq. (4.1) imposes a natural upper cutoff on the p integration at $p = O(k)$. Thus, for $0.1k < \kappa < k$, the effect of the Fisher–Langer tail is suppressed and the linear κ dependence predicted by Ward's identity approach is a satisfactory approximation.

The convolution integration in Eq. (4.1) is carried out under the "conservation of momentum" constraint, Eq. (2.14). The term $\gamma(k, \kappa)$ in the denorminator results from the frequency dependence of the relaxation rate[5] and is required for self-consistency. Except for this detail, Eq. (4.1) is identical to the integral equation employed by Résibois and Piette[(4)] for their numerical solution by iteration. Substituting Eq. (2.3), setting $\kappa = 0$ throughout, and abbreviating $\gamma(k, 0)$ by γ_k, etc., reduces Eq. (4.1) at the Curie point to

$$\gamma_k = \frac{k^2}{8\pi^3} \int \frac{d^3p}{\gamma_k + \gamma_p + \gamma_{p'}} \left(\frac{p}{p'} - \frac{p'}{p}\right)^2 \tag{4.2}$$

This integral equation supports the self-consistent solution

$$\gamma_k = ck^x \tag{4.3}$$

where the critical exponent is, by inspection,

$$x = \frac{5}{2} \tag{4.4}$$

Substituting Eqs. (4.3) and (4.4) into Eq. (4.2) and scaling the momenta to $k = 1$ determines the coefficient in Eq. (4.3) by the integral

$$c^2 = \frac{1}{8\pi^3} \int \frac{d^3p}{1 + p^{5/2} + p'^{5/2}} \left(\frac{p}{p'} - \frac{p'}{p}\right)^2 \tag{4.5}$$

With these preliminaries behind us we can now move away from the Curie point. The mass insertions in the two lines of the self-energy graph of Fig. 1a result in two spectator graphs of the type of Fig. 1c and, according to Eq. (3.2), yield

$$b = -\frac{k}{\gamma_k} \frac{\partial \gamma(k, \kappa)}{\partial \kappa}\bigg|_{\kappa = 0}^{\text{static}} = \frac{1}{4\pi} \frac{k^5}{\gamma_k^2} = \frac{1}{4\pi c^2} \tag{4.6}$$

As explained in Section 3 above, to obtain the true linear coefficient of $\gamma(k, \kappa)$, the mass insertions are not enough; we must also take into account the κ dependence of the rates in the denominator of Eq. (4.1). We split this contribution, which is denoted by K in Eq. (3.4), into two parts. The part ensuing from the variation of $\gamma(k, \kappa)$ is the integral

$$\begin{aligned} K' &= \frac{k^2}{8\pi^3} \int \frac{d^3p}{(\gamma_k + \gamma_p + \gamma_{p'})^2} \left(\frac{p}{p'} - \frac{p'}{p}\right)^2 \\ &= \frac{1}{8\pi^3 c^2} \int \frac{d^3p}{(1 + p^{5/2} + p'^{5/2})^2} \left(\frac{p}{p'} - \frac{p'}{p}\right)^2 \end{aligned} \tag{4.7}$$

[5] For a study of the deviation from true exponential relaxation at the Curie point see Ref. 10.

where the last line results from substituting Eq. (4.3) and scaling the momenta to $k=1$. Substituting c^2 from Eq. (4.5) permits us to rewrite Eq. (4.7) as

$$K' = \left\langle \frac{1}{1 + p^{5/2} + p'^{5/2}} \right\rangle \tag{4.8}$$

where the angular brackets denote an average to be computed using the integrand of Eq. (4.5) as the weighting function. It evidently follows that $0 < K' < 1$. The remaining part of K ensues from the variations of $\gamma(p, \kappa)$ and $\gamma(p', \kappa)$ in the denominator of Eq. (4.1), and is given by the average

$$K'' = \left\langle \frac{p^{3/2} + p'^{3/2}}{1 + p^{5/2} + p'^{5/2}} \right\rangle \tag{4.9}$$

Substitution of Eqs. (4.6), (4.8), and (4.9) into Eq. (3.5) gives the desired linear coefficient as

$$a = \frac{b}{1 + K' + K''} \tag{4.10}$$

We Emphasize that the required three integrations are to be carried out strictly at the Curie point, for $\kappa = 0$.

The numerical result[(3)] for Eq. (4.10) does not differ greatly from the value $a = 4/\pi = 1.27$ of Eq. (2.12). Therefore, for the present qualitative purpose of illustrating the application of Ward's identity method to the ferromagnet, we use this value as being close enough. Furthermore, we forego the self-consistent determination[(3)] of the $y \to \infty$ limit of the scaling function, as this would take us too far afield. Instead, we use the rule of thumb as a rough indication of what the self-consistent treatment would give. Consequently, we can regard the approximant $\tilde{S}(y)$ of Eqs. (A2) and (A3) and Fig. 3 (solid curve) as describing also the ferromagnet. Compared with the exact numerical computation,[6] the asymptotic value of $\tilde{S}(\infty) = 0.36$ is approximately 16% too low. Although we could remove this error by adjusting the parameters of the approximant, this is not required by our present limited goal of providing a qualitative illustration of the method.

In comparing theory with the experimental data, Als-Nielsen[11] has drawn attention to the essential role played by the sharp decrease of $\tilde{S}(y)$ in the region $0 < y < 0.5$. This effect is precisely what we have been discussing here and its idealized behavior is illustrated by the slanting

[6] The work of Ref. 3 yields $S(\infty) = 0.43$.

dashed line in Fig. 3. This linear drop dominates the actual behavior of $\tilde{S}(y)$ over almost the entire low-y range, $0 < y < 0.5$. Because the spin relaxation process in the isotropic ferromagnet is a diffusion of a conserved order parameter, its dynamics is best described in terms of the nonlocal diffusion coefficient

$$D(k, \kappa) = k^{-2}\gamma(k, \kappa) \tag{4.11}$$

At the critical point, the substitution of Eqs. (4.3) and (4.4) gives

$$D(k, 0) \propto k^{x-2} = k^{1/2} \tag{4.12}$$

At an arbitrary point in the κ–k plane the factor of $k^{1/2}$ has to be generalized to $R^{1/2}$, where R is the "distance" defined in Eq. (2.17). Thus

$$D(k, \kappa) \propto R^{1/2} S(y) \tag{4.13}$$

with the scaling function $S(y)$ depending only on the angle in the κ–k plane and being represented by an approximant $\tilde{S}(y)$ such as that plotted in Fig. 3. In our view this is the most useful and convenient way of representing $D(k, \kappa)$, because of the simple monotonic drop of $\tilde{S}(y)$ toward its $y \to \infty$ limit of $\tilde{S}(\infty)$. It is, however, customary to define a different dynamic scaling function by

$$\begin{aligned} \tilde{D}(y) &= \frac{D(k, \kappa)}{D(k, 0)} = \left(\frac{R}{k}\right)^{1/2} S(y) \\ &= (1 + y^2)^{1/4} S(y) \end{aligned} \tag{4.14}$$

$\tilde{D}(y)$ is plotted vs. y in Fig. 4, by approximating $S(y)$ by $\tilde{S}(y)$ of Eqs. (A2) and (A3) and the solid curve in Fig. 3. The $0 < y \ll 1$ and $y \gg 1$ limiting forms are shown in Fig. 4 by the dashed line and dashed curve, respectively. The factor $(R/k)^{1/2}\tilde{S}(\infty)$ is plotted as the dotted curve in Fig. 4.

The competition between the monotonic decrease in $\tilde{S}(y)$ and the monotonic increase in $(R/y)^{1/2}$ produces a minimum in $\tilde{D}(y)$ at $y = 1.1$, at which point $\tilde{D} = 0.51$. $\tilde{D}(y)$ regains the $\tilde{D}(0) = 1$ value at $y = 7.3$. Adjusting the approximant for the correct $S(\infty)$ limit (see footnote 6) (i.e., raising the underlying dashed straight line in Fig. 3 and the underlying dashed curve in Fig. 4) will bring the $\tilde{D}(y) = 1$ crossing down to $y = 5.3$ and will bring up the minimum value into good agreement with the value 0.57 reported by Résibois and Riette.[4] These authors described the minimum as a "nontrivial" result. From the present point of view it will be clear that the minimum is a direct and immediate consequence of the linear drop in $D(k, \kappa)$ in the neighborhood of $\kappa = 0$. This linear drop follows very simply, in turn, from the application of Ward's identity, as described in this paper.

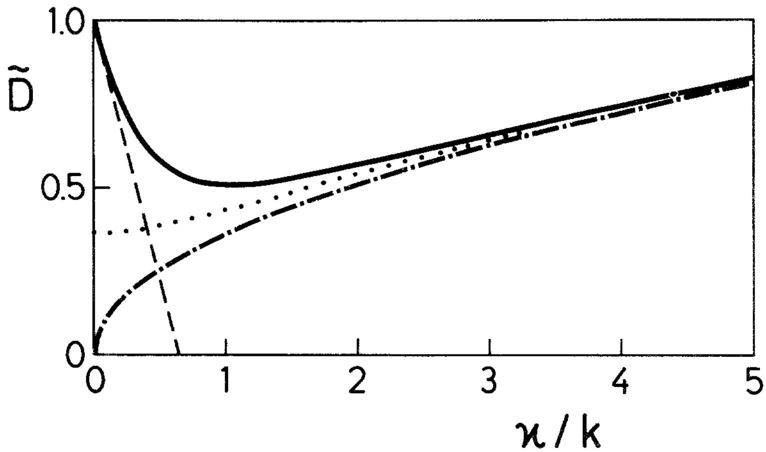

Fig. 4. Conventional ferromagnet scaling function, $\tilde{D}$ [see Eq. (4.14)] vs. κ/k, the reciprocial of the product of wave number times correlation length. The solid curve is a schematic representation obtained by multiplying $\tilde{S}$ of Fig. 3 by the "radial" factor $(R/k)^{1/2} = (1 + \kappa^2/k^2)^{1/4}$ (shown by the dotted curve). The competition between the rise in this factor and the initial linear drop from Ward's identity (slanting dashed line) produces the minimum at $\kappa/k \simeq 1.1$. Because of its monotonicity, the scaling function S as defined in Eq. (4.13) and as illustrated in Fig. 3 is preferred over the conventional definition $\tilde{D}$, as illustrated here by the solid curve. The dot–dash curve shows the $\kappa \gg k$ hydrodynamic behavior of $\tilde{D}$ and, in a more exact calculation, lies somewhat higher.

5. SUMMARY

Taking the simple polarization function $C(k, \kappa)$ as a prototype, we separated its dependence on its two variables by factoring off R^{-1}, where R is the "distance" in the κ plane of Fig. 1. What remains is the scaling function $S(y)$, where $y = \kappa/k$ is the "angle" variable. The detailed and special structure of $C(k, \kappa)$ is contained in $S(y)$. This definition of the scaling function eliminates the gross changes in $C(k, \kappa)$ that come from R and leaves $S(y)$ to vary (monotonically) between the two limits $S(0)$ and $S(\infty)$, as shown in Fig. 3. These limiting values differ merely by a numerical factor of order 1. We then saw that this definition of $S(y)$ brings with it the substantial advantage that most of the variation of $S(y)$ between its two limits is associated with the linear drop, that takes place in the region $0 < y < 0.5$. This linear drop, shown by the slanting dashed line in Fig. 3, is the salient feature of $S(y)$ and is the central theme of this paper. For larger values of y, $S(y)$ approaches its limiting value, $S(\infty)$, very slowly (as y^{-2}). This subsequent variation takes us outside the region of applicability of Ward's identity approach, but it is easily and effectively handled by means of a Padé-type approximant. Thus it is possible in any given case to give a good representation of $S(y)$ by means of a two-parameter approximant, $\tilde{S}(y)$.

The representation in Appendix A of the scaling function for $C(k, \kappa)$ by its two-parameter approximant sets the stage for the corresponding treatment of a dynamic scaling function. In Section 3 we dealt with the "feedback effect' of the relaxation rates on their $0 < y \ll 1$ behavior. This effect does not constitute any essential complication of Ward's identity method and is, in general, taken into account by the inversion of an appropriate matrix. As a concrete illustration of the method, we sketched the theory of the dynamic scaling function for the isotropic Heisenberg ferromagnet in Section 4. As expected, and as illustrated by the slanting dashed line in Fig. 4, the initial linear drop in $\tilde{D}(y)$ dominates its behavior in the region $0 < y < 0.5$. The increase in $\tilde{D}(y)$ for $y \gg 1$ is a consequence of the way $\tilde{D}(y)$ has been defined by Résibois and Piette.[4] The fact that the minimum in $\tilde{D}(y)$ at $y \simeq 1$ has this simple and natural origin demonstrates the usefulness of Ward's identity method that we have presented in this paper.

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APPENDIX A: APPROXIMATE SCALING FUNCTION

In order to demonstrate the full usefulness of Ward's identity method we discuss here a simple method for obtaining an approximation $\tilde{S}(y)$ to the true scaling function $S(y)$, as a function of $y = \kappa/k$, for the general case where a closed expression for $S(y)$, such as Eq. (2.23), is not available. From the underlying integral equations we deduce the following general conditions:

(a) Linear dependence on y for $0 < y \ll 1$ as expressed by Eq. (2.24).

(b) Monotonic decrease of $\tilde{S}(y)$ between the two limits $\tilde{S}(0)$ and $\tilde{S}(\infty)$. (This may be difficult to establish rigorously, in which case it may be adopted as a reasonable working hypothesis.)

(c) Asymptotic approach to the $y \to \infty$ limit as

$$\tilde{S}(y) - \tilde{S}(\infty) \propto y^{-2} \tag{A1}$$

This property is exhibitly explicitly by the integral in Eq. (2.7) and follows from the general requirement that the k dependence of such integrals can only be in terms of k^2, in the range $k \ll \kappa$. In order to find an approximant that satisfies the above three general conditions we extend Eq. (2.24) by replacing the linear factor $y \equiv \kappa/k$ by some function $f(m\kappa/k) = f(my)$ where, by the rule of thumb, we expect $m \simeq 2$. $f(my)$ should have an initial linear rise and then saturate at some finite limiting value, $f(\infty)$, as $y \to \infty$. If we normalize the initial slope as $f'(0) = 1$ we can write the approximant as

$$\tilde{S}(y) = 1 - \frac{a}{m} f(my) \tag{A2}$$

An especially simple form possessing the right properties is

$$f(z) = \frac{z}{(1 + z^2)^{1/2}} \tag{A3}$$

In addition to $f'(0) = 1$, Eq. (A3) gives the desired saturation at $f(\infty) = 1$ and the approach to this limiting value for $z \gg 1$ as

$$f(z) \simeq 1 - \frac{1}{2} z^{-2} \tag{A4}$$

Thus, all three of the above general requirements, (a), (b), and (c), are satisfied. In general, it is possible to compute an approximate value for $\tilde{S}(\infty)$, which, with

$$\tilde{S}(\infty) = 1 - \frac{a}{m} \tag{A5}$$

and the already-determined value of a fixes m. But we do not want to go into this matter in such detail here. For demonstrating the general idea, it suffices to set $m = 2$, its rule-of-thumb value, so that Eq. (A5) gives $\tilde{S}(\infty) = 1 - 2/\pi = 0.363$. This is somewhat larger than the correct value of $S(\infty) = \pi^{-1} = 0.318$, which is illustrated in Fig. 3 by the horizontal dashed line. The approximant of Eqs. (A2) and (A3) for $m = 2$ is plotted vs. $y = \kappa/k$ as the solid curve in Fig. 3. This approximant serves our present purpose, which is to show how the linear part of $S(y)$, which is the main point of our paper, can be easily supplemented by a Padé-type treatment so as to obtain a reasonably good representation of $S(y)$ for the entire range $0 \leqslant y \leqslant \infty$. Although we do not exhibit it here, it is clear that an even better fit can be obtained by adjusting the second parameter of the approximant. Taking $m = 4/(\pi - 1) = 1.87$ would, according to Eq. (A5) make $S(y)$ exact in the $y \to \infty$ limit as well as in the small-y region.

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